

# A Mixed-Boolean Hybrid Mathematical Model with Discontinuous States

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**Abstract:** Hybrid mathematical models are often represented as continuous functions with discontinuous inputs, or they are visualised as state machines or petri-nets comprising continuous models linked by discontinuous mappings. The analysis and simulation of hybrid (or nonsmooth dynamical) models is plagued with difficulty, necessitating careful consideration of energy losses and state reinitialisation on commutation. The author proposes an alternative model, where states are discontinuous. The engineer familiar with techniques such as signal flow graphs or bond graphs can clearly visualise discontinuities as breaks (or joins) in power flow between parts of the model. A mixed-Boolean state equation can be derived which reflects the physics of switching behaviour. This has two advantages: first, by considering the physics incrementally about the discontinuity it can be simulated without the need for state reinitialisation algorithms, and second, it can be analysed for structural control properties to show how they change with commutation.

*Keywords:* Hybrid models, State space methods, Switching variables, Systems models, Bond graphs.

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## 1. INTRODUCTION

Hybrid models are those containing continuous and discontinuous functions. They are used to model variable structure systems (such as contact), and those where rapidly changing nonlinear behaviour can be described by some form of piecewise continuous equation (such as stiction/friction).

Hybrid systems can be visualised as continuous modes on areas of state space linked by a discontinuous state mapping (Mosterman et al., 1998) and described as a hybrid automaton i.e. one that contains both finite and continuous state spaces (Van Der Schaft and Schumacher, 1999). The dynamics consist of discrete transitions plus an evolution of the continuous part in each location.

There are many ways of abstracting a physical system to a hybrid model and, it appears that the diversity of methods reflect the many applications and tools available.

Sliding Mode Control for Variable Structure Systems assumes systems “governed by ordinary differential equations with discontinuous state functions in the right hand side” (i.e. the input) (Utkin, 1992). It is well established that a discontinuous control action typically in the form of a switching input causes the systems structure to vary. A subspace or hyperplane (the switching surface) divides the state space of the model into two regions, each with a different control law (or form of). When the system operates on the switching surface, it is said to be in sliding mode and sliding control utilises this idea to give robust control in discontinuous and nonlinear systems. This method can be extended to variable-structure systems where the parameters - and not just the control inputs - are discontinuous (Marghitu and Irwin, 2001).

Petri-nets can be used to describe a set of interlinked continuous models, but these can become large:  $2^n$  models where  $n$  is the number of switches (Borutsky, 1995).

The linear complementarity problem, comprising a continuous equation (such as a state equation) and complementarity condition (Van Der Schaft and Schumacher, 1999) is perhaps the most widely used model in the field of nonsmooth dynamics. These contain an external signal which can be thought of as a Lagrange multiplier, and commute between zero and a value which must be calculated. This model can be transferred to a single inclusion or a variational inequality, which have unique continuous solutions.

Mixed logical dynamical (MLD) systems i.e. those with interdependent physical laws, logic rules, and operating constraints, have been established (Bemporad and Morari, 1999) and shown to be equivalent to other classes of hybrid system (Heemels et al., 2001). The model presented here differs in that it originates from idealised physical modelling (i.e. the Bond Graph) and explicitly embraces the physics and changing causality of the system.

Users of commercial-off-the-shelf software can naively use ‘switches’ in a model without appreciating the impact this has. In the author’s experience this frequently happens, with an inexperienced engineer or researcher left attempting to tackle algebraic loops or integration errors by randomly inserting transfer functions and sources of compliance before a pressing deadline. Clearly, this is an unacceptable strategy. The author’s work on Bond Graph methodology was motivated by the need to promote deep understanding of both physics and computational considerations among analysts, and informs this research.

Willems presents a case for using idealised physical modelling methods (of which bond graphs are an example) to mitigate against unwittingly creating physically meaningless or computationally inefficient models (Willems, 2007).

The author argues that, in cases where the discontinuity is an integral part of the physical system (a switch, mechanical contact, or some highly nonlinear behaviour abstracted by the modeller to a heaviside function), it is more physically representative for the state variables to be discontinuous.

## 2. CATEGORISATION OF DISCONTINUITIES

Branicky et al (Branicky et al., 1998) categorise hybrid models into *Switching* and *Impulse* models, which can be *controlled* or *autonomous*.

Switching models are defined as those where the vector field changes discontinuously when the state hits a boundary. Switching systems “comprise a family of dynamical subsystems together with a switching signal determining the active system at a current time” (Vu and Liberzon, 2008). They are a subset of hybrid systems, where there is some discontinuous behaviour modelled by an on/off switch or other binary signal.

Impulse models are those where the continuous state changes impulsively on hitting prescribed regions of state space. The classic example is Newtons Collision law, where the state of a body changes from positive to negative velocity on impact, and any dissipative effects are accounted for by a coefficient of restitution. The state changes impulsively, and there is an impulse loss on commutation.

The author proposes a further distinction between *Structural* and *Parametric* discontinuities (Margetts, 2013).

*Proposition 1.* Structural Discontinuities occur when parts of the model are connected or disconnected, interrupting power flow between components. These discontinuities often give rise to variable topology models.

Engineering examples of structural discontinuities are the hydraulic valve, mechanical clutch, ideal electrical switch, or contact between bodies.

*Proposition 2.* Parametric Discontinuities occur when an element has a highly nonlinear constitutive equation, which has been abstracted to a piecewise continuous function. The structure of the model is unchanged, it is the equation describing the behaviour of an element which changes.

Common examples of parametric discontinuities are dry friction, tyre forces, a two-stage oleo strut ‘breaking out,’ or saturation of an electrical capacitor or hydraulic accumulator.

## 3. DERIVING THE MIXED-BOOLEAN MODEL

Bond Graphs were instrumental in deriving the mixed-Boolean model. Bond Graphs are an idealised physical modelling method, enabling the user to sketch a system, assign computational causality, and derive a state space model. Readers unfamiliar with the technique are directed to Karnopp, Margolis and Rosenberg’s standard text (Karnopp et al., 2006).

There has been a significant body of work on Hybrid Bond Graphs, with numerous variations proposed and extensive discussion of their implications for structural analysis and simulation. The author’s work grew from a desire to construct hybrid models which accurately reflect the physics of the system – offering the user insight – as well as being suitable for accurate simulation. Previously, none of the proposed Hybrid Bond Graphs achieved this. A thorough literature review is given in (Margetts et al., 2013).

A form of Hybrid Bond Graph can be defined which incorporates ‘controlled junctions’ for structural discontinuities (Margetts et al., 2013) and ‘controlled elements’ (containing a mode-switching ‘tree’ of junctions and elements) for parametric discontinuities (Margetts and Ngwompo, 2015). The control signals are assigned a Boolean parameter, which sets each junction to an ON or OFF state (1 or 0). When ON, power flows through the junction uninhibited. When OFF, the junction is effectively replaced with null sources or sinks.

Equations relating the elements in the system can hence be written in terms of Boolean parameters. The equations fully describe the system in all potential modes of operation: when Boolean parameters are ‘ON’ or ‘true,’ the equations are multiplied by one, but when they are ‘OFF’ or ‘false’ the equations are multiplied by zero and cease to be part of the model for that mode of operation.

One important point highlighted by use of the hybrid bond graph is the presence of *dynamic causality*. Bond graphs are an acausal modelling method i.e. the model is constructed *before* inputs and outputs are defined. This means that the state space model derived from the bond graph can be put in ‘preferred integral causality’ so as to aid computation. However, in hybrid models the ideal ‘preferred integral causality’ assignment can change on commutation. Many users seek to constrain dynamic causality, usually by adding sources of compliance or ‘causality resistance’ (Asher, 1993; Breedveld, 2000, 2002). This is somewhat controversial as it can yield stiff models and slow simulation times, while disregarding important physical information (Cellier et al., 1994; Buisson, 1993). For example, in the case of rigid contact the dynamic causality reflects the genuine kinematic constraint between two bodies (which can temporarily be considered as a single rigid body).

For the Hybrid Bond Graph. The same rules for deriving a state space model from a regular bond graph are followed. This derivation revolves around turning the graphical bond graph model into a ‘Junction Structure Matrix.’ This is a matrix of 1’s and 0’s which link all of the inputs and outputs in the model. The difference with the Hybrid bond graph is that a Boolean term can be inserted. For parametric discontinuities, Boolean terms are contained in the expression for an element. For structural discontinuities, Boolean terms are introduced to the Junction Structure Matrix where the relationship between inputs and outputs depends on commutation. The matrix equation is then rearranged into state space form, by placing it in terms of the states (which are the inputs to the storage elements). A generalised Junction Structure Matrix is shown in equation (1). Fig. 1 shows the key

variables used in the causally dynamic hybrid bond graph as inputs to and outputs from this junction structure.

Note that dynamic causality is handled in the hybrid bond graph by assigning two inputs and two outputs to the affected elements: in each mode of operation one input and output is ‘true’ and the other is removed from the model by multiplying it by zero. When a state space model is derived, the order will reflect the potential states in any mode, and the model may be implicit.

$$\begin{bmatrix} \mathbf{\Lambda}_{11} & 0 & 0 \\ 0 & \mathbf{\Lambda}_{22} & 0 \\ 0 & 0 & \mathbf{\Lambda}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{X}}_i \\ \mathbf{Z}_d \\ \mathbf{D}_{out} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13} & \mathbf{S}_{14} \\ -\mathbf{S}_{12}^T & 0 & 0 & \mathbf{S}_{24} \\ -\mathbf{S}_{13}^T & 0 & \mathbf{S}_{33} & \mathbf{S}_{34} \end{bmatrix} \begin{bmatrix} \mathbf{Z}_i \\ \dot{\mathbf{X}}_d \\ \mathbf{D}_{in} \\ \mathbf{U} \end{bmatrix} \quad (1)$$

Here  $\mathbf{X}$  is a vector of states or – where there is derivative causality – pseudo-states. These are associated with storage elements (i.e. inertias and compliance). Subscripts  $i$  and  $d$  denote integral and derivative causality respectively, and the circumflex and tilde denote static and dynamic causality respectively.  $\mathbf{Z}$  is a complementary vector of a state vector.  $\mathbf{U}$  is the vector of inputs and  $\mathbf{V}$  is its complementary vector.  $\mathbf{D}$  describes the input and output associated with dissipative elements (subscript *in* and *out*). Finally,  $\mathbf{S}$  are submatrices of the Junction Structure Matrix, composed of 1’s, 0’s, Boolean parameters  $\lambda$ , and coefficients associated with Transformer and Gyrator elements  $TF$  and  $GY$ . It is worth noting that an element can only have two modes of operation (flow input / effort output and effort input / flow output), although a model can have several modes of operation overall if it contains multiple controlled junctions.

There are two inputs and two outputs for each 1-port element in dynamic causality. The input/output sets are exclusive of each other, and the Boolean terms in the Hybrid Junction Structure Matrix will activate one of these for each mode of operation.

In order to establish which outputs of the junction structure are active, the vector of outputs must be multiplied by a diagonal matrix of Boolean expressions  $\Lambda(\lambda)$ . In any one mode of operation, some rows of the matrices will be set to zeros and others will give the Junction Structure for that mode. Therefore, outputs which are in static causality are assigned a 1 in the diagonal of the matrix  $\Lambda(\lambda)$  because they are fixed outputs, while variables associated with elements in dynamic causality are assigned a Boolean function  $f(\lambda)$  determined by the combination of the switch parameters  $\lambda$  that dictate the output status of the variable. For each Boolean term  $f(\lambda)$ , there will always be a NOT term  $\overline{f(\lambda)}$  present in the matrix  $\Lambda(\lambda)$  which activates another row to describe the dynamic elements behaviour in its other state.

In general, the mixed-Boolean state equation takes the form:

$$\mathbf{\Lambda} \dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}, \mathbf{z}, \mathbf{u}, \lambda, t) \quad (2)$$

$$\mathbf{0} = \mathbf{f}_2(\mathbf{x}, \mathbf{z}, \mathbf{u}, \lambda, t) \quad (3)$$

Where  $\mathbf{x}$  is the state vector,  $\mathbf{z}$  is a vector of semi-state variables,  $\mathbf{u}$  is the input vector,  $\lambda$  is a vector of the Boolean variables associated with switching, and  $\mathbf{\Lambda}$  is a matrix of Boolean terms which activate/deactivate states according to the mode of operation. For Linear Time-Invariant systems, the model becomes:

$$\mathbf{E}(\mathbf{\Lambda}) \dot{\mathbf{x}} = \mathbf{A}(\lambda) \mathbf{x} + \mathbf{B}(\lambda) \mathbf{u} \quad (4)$$

#### 4. STRUCTURAL ANALYSIS

A major advantage of the mixed-Boolean state model is that a user can see how its properties change with commutation.

Consider the control properties frequently obtained from LTI State Space models, such as solvability, stability, controllability, and observability. Where Boolean terms occur in the state matrix, they can set a row or column to zeros or finite values depending on the mode of operation. This means that the rank and order of the model can change with commutation. Since solvability, controllability and observability are usually established from the rank of the state matrix, they consequently change also.

This reflects the physics the systems. An electrical switch or mechanical clutch might disconnect two subsystems when ‘OFF,’ resulting in an uncontrollable, unobservable subsystem. When two parts make rigid contact, a kinematic constraint might be formed with obvious consequences for solvability.

Stability and dynamic modes can also be established from the state matrix. The presence of Boolean terms in the state matrix means that eigenvalues will change with commutation. Again, this reflects the physics of the system: as elements are connected and disconnected, the inertias, compliance and dissipation in each subsystem is altered, and hence the dynamic behaviour will change. A falling body clearly has different dynamics to one in contact with the ground.

#### 5. SIMULATION

Hybrid models are typically problematic with regards to simulation. A number of issues merit consideration, as outlined by Acary and Brogliato (Acary and Brogliato, 2008).

- Switching must always occur at the end of a time-step, in order to be captured. This usually motivates an event-driven method. However, this can be impractical where there is a large number of switching instants or it is not known where they occur.
- Chattering may occur where a sliding mode cannot be reached due to numerical approximation.
- A procedure for accurately finding the location of events may be required, along with some method for reinitialising states after the event.
- Where there are a number of events, there may be a finite accumulation point past which the event-driven method cannot progress.
- There may be an impulsive term on commutation giving a *Dirac* or *Stieltjes* measure. For example, the differential measure of velocity which manifests on impact between bodies.

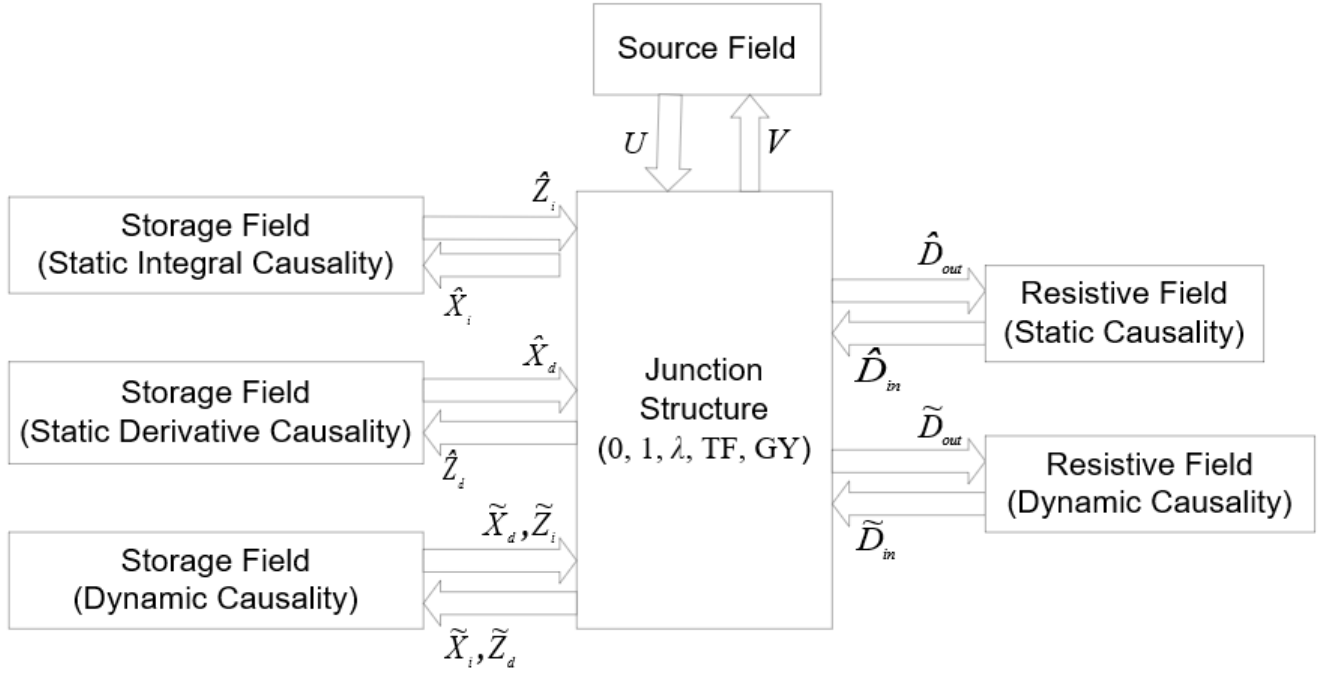


Fig. 1. Quantities used in Hybrid Junction Structure Matrix and Subsequent Development

$$\begin{bmatrix} (\lambda_1 \oplus \lambda_2) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda_3 \\ 0 & 0 & \bar{\lambda}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_1 d \\ \dot{p}_3 d \end{bmatrix} = \begin{bmatrix} \frac{-(\lambda_1 \oplus \lambda_2)}{L_1 R_1} & \frac{-a(\lambda_1 \oplus \lambda_2)}{L_2} & 0 & 0 & 0 \\ \frac{a(\lambda_1 \oplus \lambda_2)}{L_1} & \frac{L_2}{\lambda_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{\lambda}_3}{L_3 R_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(\lambda_1 \oplus \bar{\lambda}_2)}{L_1} & 0 \\ 0 & \frac{\lambda_3}{L_2} & 0 & 0 & -\frac{\lambda_3}{L_3} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_1 d \\ p_3 d \end{bmatrix} + \begin{bmatrix} \lambda_1 & \lambda_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ G \end{bmatrix} \quad (5)$$

The mixed-Boolean model does present a partial solution. The Hybrid Bond Graph it is derived from is a *Switched* model, not an impulsive one. It includes all modes of operation – no matter how short – hence there is no impulse loss. Since there is no impulse loss, states do need to be reinitialised: they can simply be calculated from their value prior to commutation.

This, however, presents another problem. Hybrid models are often used to avoid unnecessarily stiff dynamic models which are slow to simulate. Including a negligibly short mode of operation (for example, the ‘in contact with the ground’ phase of a bouncing ball) necessitates an extremely stiff model, with small time-steps required to capture the short mode. This is why collision problems are usually abstracted to impulse models. In these cases, the mixed-Boolean [switching] state model can be used to derive a coefficient of restitution, and generate an impulse model (Margetts and Ngwompo, 2014).

Where events occur at a given time, it is relatively trivial to simulate the mixed-Boolean model. It can be modelled in commercial software such as MATLAB<sup>®</sup> using the symbolic math toolbox (Margetts et al., 2017). Values are assigned to the parameters (including the Boolean ones) and the model is simulated up until the event time. New Boolean parameters are then assigned, and

the simulation continues. If the model is implicit in some modes of operation, an implicit solver must be used: these are now commonplace in most commercial modelling and simulation packages and work well with models of this type (Boudon et al., 2015). Alternatively, the model could be simplified in each individual mode of operation (e.g. combining rigid bodies which are in contact), although this could represent significant work in a complex model unless automated.

Likewise, a Matlab script can be written which halts simulation and reassigns Boolean variables when some event occurs, by checking whether some state has crossed a threshold value. This strategy can however be problematic: the location of the event may not be accurate, and accumulation of events has not been addressed. Chattering can still occur. Future work could therefore utilise techniques such as discrete-time Moreau’s second-order sweeping process, where there are no detection times and hence no accumulation point (Acary and Brogliato, 2008).

## 6. CASE STUDY

A simple case study is presented to illustrate the mixed-Boolean model. Consider a power converter supplying a DC Motor, which drives a rotating load via a clutch. The



This case study is explored in more depth in reference (Margetts et al., 2017).

## 7. CONCLUSIONS

By considering the physics of discontinuities and nonsmooth dynamics, a mixed-Boolean state equation is obtained. This novel model is discontinuous on the states, rather than on the input. The model sidesteps some of the issues with simulating hybrid systems, namely the need to reinitialise states after events. In addition to being suitable for simulation using common commercial packages, the model can be used for structural analysis. The Boolean terms clearly show how the system's dynamics and control properties can change with commutation.

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## REFERENCES

- Acary, V. and Brogliato, B. (2008). Numerical Methods for Nonsmooth Dynamical Systems: Applications in Mechanics and Electronics.
- Asher, G.M. (1993). The Robust Modelling of Variable Topology Circuits Using Bond Graphs. In *Proc. 1993 Western Simulation Multiconference - Int. Conf. on Bond Graph Modeling ICBGM'93*, 126–131. La Jolla, CA.
- Bemporad, A. and Morari, M. (1999). Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3), 407 – 427. doi: [http://dx.doi.org/10.1016/S0005-1098\(98\)00178-2](http://dx.doi.org/10.1016/S0005-1098(98)00178-2). URL <http://www.sciencedirect.com/science/article/pii/S0005109898001782>.
- Borutzky, W. (1995). Discontinuities in a bond graph framework. *J. Franklin Institute - Eng. and Appl. Math.*, 332B, 141–154.
- Boudon, B., Margetts, R., Borutzky, W., and Malburet, F. (2015). Methodology and numerical comparison of simulation methods of multi-body systems with bond graphs. *Unpublished*.
- Branicky, M.S., Borkar, V.S., and Mitter, S.K. (1998). A Unified Framework for Hybrid Control: Model and Optimal Control Theory. *IEEE Trans. Autom. Control*, 43, 31–45. doi:10.1109/9.654885.
- Breedveld, P.C. (2000). An alternative Model for Static and Dynamic Friction in Dynamic System Simulation. In *1st IFAC Conf. Mechatronic Systems*, 717–722. Darmstadt, Germany.
- Breedveld, P.C. (2002). Modelling & Simulation Of Bouncing Objects: Newton's Cradle Revisited. In *Mechatronics 2002*. Twente, The Netherlands.
- Buisson, J. (1993). Analysis and Characterization of Hybrid Systems With Bond-Graphs. In *Proc. Int. Conf. on Systems, Man and Cybernetics: Systems Engineering In the Service of Humans*, volume 1, 264–269. IEEE, Le Touquet, France.
- Cellier, F.E., Otter, M., and Elmqvist, H. (1994). *Simulation Series*, 27, 49.
- Heemels, W.P., De Schutter, B., and Bemporad, A. (2001). Equivalence of hybrid dynamical models. *Automatica*, 37(7), 1085–1091.
- Karnopp, D.C., Margolis, D.L., and Rosenberg, R.C. (2006). *System Dynamics Modeling and Simulation of Mechatronic Systems*. John Wiley & Sons, Inc., Hoboken, N.J., fourth edition.
- Margetts, R. and Ngwompo, R.F. (2014). Hybrid bond graphs for contact, using controlled junctions and dynamic causality. In *Proceedings of the Summer Simulation Multiconference 2014*. SCS, Monterey, CA.
- Margetts, R., Ngwompo, R.F., and Fortes da Cruz, M. (2013). Construction and Analysis of Causally Dynamic Hybrid Bond Graphs. *Proc. IMechE Part I - J. Syst. and Control Eng.*, 227, 329–346.
- Margetts, R. and Ngwompo, R. (2015). Mode switching in causally dynamic hybrid bond graphs. *Mechatronics*. doi:10.1016/j.mechatronics.2015.04.013.
- Margetts, R. (2013). *Modelling & Analysis of Hybrid Dynamic Systems using a Bond Graph Approach*. Ph.D. thesis, University of Bath, Bath, UK.
- Margetts, R., Ngwompo, R.F., and Boudon, B. (2017). Simulation of Causally Dynamic Hybrid Bond Graphs, with Application to a Power Converter. In *Unpublished*.
- Marghitu, D. and Irwin, J. (2001). *Mechanical Engineer's Handbook*. Elsevier Academic Press.
- Mosterman, P.J., Zhao, F., and Biswas, G. (1998). An Ontology for Transitions in Physical Dynamic Systems. In *15th Nat. Conf. on Artificial Intelligence (AAAI 98)*, 219–224. American Assoc. Artificial Intelligence, Madison, WI.
- Ogata, K. (2010). *Modern control engineering*. Prentice-Hall.
- Utkin, V.I. (1992). *Sliding Modes in Control Optimization*. Springer-Verlag.
- Van Der Schaft, A.J. and Schumacher, J.M. (1999). *An Introduction to Hybrid Dynamical Systems*, volume 251. Springer, London.
- Vu, L. and Liberzon, D. (2008). Invertibility of switched linear systems. *Automatica*, 44, 949–958.
- Willems, J.C. (2007). The Behavioural Approach to Open and Interconnected Systems. *IEEE Control Syst. Mag.*